

INEQUIVALENT CONTACT STRUCTURES ON BOOTHBY-WANG 5-MANIFOLDS

M. J. D. HAMILTON

ABSTRACT. We consider contact structures on simply-connected 5-manifolds which arise as circle bundles over simply-connected symplectic 4-manifolds and show that invariants from contact homology are related to the divisibility of the canonical class of the symplectic structure. As an application we find new examples of inequivalent contact structures in the same equivalence class of almost contact structures with non-zero first Chern class.

CONTENTS

1. Introduction	1
2. Classification of simply-connected 5-manifolds	2
3. Contact structures on simply-connected 5-manifolds	4
4. Topology of circle bundles	7
5. The Boothby-Wang construction	11
6. The construction for symplectic 4-manifolds	13
7. Contact homology	14
8. Applications	21
References	25

1. INTRODUCTION

The Boothby-Wang construction [2] associates to each symplectic manifold (M, ω) , such that the symplectic form ω represents an integral class in $H^2(M; \mathbb{R})$, a contact structure ξ on the circle bundle X over M whose Euler class is given by the class represented by ω . In this article we are interested in the case where X is a simply-connected closed 5-manifold. In Section 4 we will show that in this case the 4-manifold M also has to be simply-connected and the Euler class $[\omega]$ indivisible. In addition, it follows that the integral homology of X is torsion free. By the classification of simply-connected closed 5-manifolds due to D. Barden [1], it is possible to determine the 5-manifold X up to diffeomorphism: X is diffeomorphic either to the connected sum

$$\#(b_2(M) - 1)S^2 \times S^3$$

Date: January 12, 2010.

2000 Mathematics Subject Classification. 53D10, 53D05, 53D42.

Key words and phrases. contact structure, 5-manifold, symplectic, 4-manifold, canonical class.

or to

$$\#(b_2(M) - 2)S^2 \times S^3 \# S^2 \tilde{\times} S^3$$

depending on whether X is spin or non-spin. Here $S^2 \tilde{\times} S^3$ denotes the non-trivial S^3 -bundle over S^2 . Moreover, the 5-manifold X is spin if and only if M is spin or the mod 2 reduction of the Euler class $[\omega]$ equal to the second Stiefel-Whitney class of M . As a consequence of this diffeomorphism classification, one can construct Boothby-Wang contact structures on the same simply-connected 5-manifold X using different simply-connected symplectic 4-manifolds (M, ω) and (M', ω') . Up to the spin condition, the 4-manifolds only need to have the same second Betti number.

In Section 3 we consider contact structures and almost contact structures on simply-connected 5-manifolds in general. In particular, we consider the notion of *equivalence* of these structures, i.e. when two such structures can be made identical by a sequence of deformations and self-diffeomorphisms of the manifold. We will show that two almost contact structures are equivalent on a simply-connected 5-manifold if and only if their first Chern classes have the same maximal divisibility.

Since symplectic 4-manifolds exist in great number, it is likely that many of the induced Boothby-Wang contact structures on the same 5-manifold X are not equivalent as contact structures, even if they are equivalent as almost contact structures. In Section 7 we will show that invariants derived from contact homology defined in [5] are related to the divisibility of the canonical class of the symplectic structure on the simply-connected 4-manifold. This is summarized in the main result Corollary 43. It shows that the existence of inequivalent contact structures on simply-connected 5-manifolds with torsion free homology is connected to the geography question of simply-connected 4-manifolds with divisible canonical class. As an application we find new examples of inequivalent contact structures in the same equivalence class of almost contact structures with non-zero first Chern class. A related discussion has appeared in [18]. Inequivalent contact structures on simply-connected 5-manifolds with vanishing first Chern class have been found before by O. van Koert in [13]. Also I. Ustilovsky [19] found infinitely many contact structures on the sphere S^5 and F. Bourgeois [3] on $T^2 \times S^3$ and T^5 , both in the case of vanishing first Chern class.

Acknowledgements. The content of this article is part of the author's Ph.D. thesis, submitted in May 2008 at the University of Munich. I would like to thank D. Kotschick for supervising the thesis and K. Cieliebak and O. van Koert for helpful comments. I am also grateful to the *Studienstiftung des deutschen Volkes* and the *Deutsche Forschungsgemeinschaft* for financial support.

2. CLASSIFICATION OF SIMPLY-CONNECTED 5-MANIFOLDS

Throughout this article, we use for a topological space Y the abbreviations $H_*(Y)$ and $H^*(Y)$ to denote the homology and cohomology groups of Y with \mathbb{Z} -coefficients. Other coefficients will be denoted explicitly.

In this section we recall the classification of simply-connected closed 5-manifolds due to D. Barden [1] and refer to this article for further details. Let X denote a

smooth, closed, oriented 5-manifold. For each pair of elements η, ξ in the torsion subgroup $\text{Tor}H_2(X)$ there exists a *linking number* $b(\eta, \xi)$ in \mathbb{Q}/\mathbb{Z} . These numbers define a skew-symmetric non-degenerate bilinear form

$$b : \text{Tor}H_2(X) \times \text{Tor}H_2(X) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

called the *linking form*. Suppose that the 5-manifold X is simply-connected. Then the first integral homology group vanishes and the Universal Coefficient Theorem implies that there exists an isomorphism

$$H^2(X; \mathbb{Z}_2) \cong \text{Hom}(H_2(X), \mathbb{Z}_2),$$

via evaluation of cohomology on homology classes. Hence we can think of the second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbb{Z}_2)$ as a homomorphism

$$w_2(X) : H_2(X) \rightarrow \mathbb{Z}_2.$$

The following theorem is the classification theorem for simply-connected 5-manifolds and was proved by Barden [1, Theorem 2.2] using surgery theory:

Theorem 1. *Let X, Y be simply-connected, closed, oriented 5-manifolds. Suppose that $\theta : H_2(X) \rightarrow H_2(Y)$ is an isomorphism preserving the linking forms on the torsion subgroups and such that $w_2(Y) \circ \theta = w_2(X)$. Then there exists an orientation preserving diffeomorphism $f : X \rightarrow Y$ such that $f_* = \theta$.*

Since the linking number and the second Stiefel-Whitney class are homotopy invariants, it follows in particular that simply-connected, closed 5-manifolds which are homotopy equivalent are already diffeomorphic.

It is possible to give a complete list of building blocks of simply-connected 5-manifolds such that each simply-connected 5-manifold is a connected sum of some of those building blocks. In the following, we are particularly interested in simply-connected 5-manifolds X whose integral homology is torsion free. By Poincaré duality and the Universal Coefficient Theorem the whole integral homology is torsion free if and only if the second homology $H_2(X)$ is torsion free. Simply-connected 5-manifolds satisfying this condition have a simple structure, because they can be constructed using only two building blocks, which can be described in the following way.

There exist up to isomorphism precisely two oriented S^3 -bundles over S^2 – the trivial bundle $S^2 \times S^3$ and a non-trivial bundle denoted by $S^2 \tilde{\times} S^3$. The manifold $S^2 \tilde{\times} S^3$ can be constructed as follows: Let $B = S^2 \tilde{\times} D^3$ denote the non-trivial D^3 -bundle over S^2 . Then the boundary ∂B is the non-trivial S^2 -bundle over S^2 , hence diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Let $\phi : \partial B \rightarrow \partial B$ denote the orientation reversing diffeomorphism obtained by interchanging the summands of ∂B . Then the 5-manifold $S^2 \tilde{\times} S^3$ is obtained by gluing together two copies of B along their boundaries via the diffeomorphism ϕ . In particular, the manifold $S^2 \tilde{\times} S^3$ is non-spin, because a spin structure would induce a spin structure on B and hence on ∂B , which is non-spin.

It follows from the list of building blocks in Barden's article [1] that $S^2 \times S^3$ and $S^2 \tilde{\times} S^3$ are the only building blocks with torsion free second integral homology. Hence every simply-connected 5-manifold with torsion free homology decomposes

as a connected sum of several copies of these two manifolds. Moreover, one can show with Theorem 1 that there exists a diffeomorphism

$$S^2 \tilde{\times} S^3 \# S^2 \tilde{\times} S^3 \cong S^2 \times S^3 \# S^2 \tilde{\times} S^3,$$

hence in every non-spin connected sum one $S^2 \tilde{\times} S^3$ summand suffices. This implies:

Proposition 2. *Let X be a simply-connected closed oriented 5-manifold with torsion free homology. Then X is diffeomorphic to*

- (a) $\#b_2(X)S^2 \times S^3$ if X is spin
- (b) $\#(b_2(X) - 1)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ if X is non-spin.

The empty sum in (a) for $b_2(X) = 0$ is the 5-sphere S^5 .

3. CONTACT STRUCTURES ON SIMPLY-CONNECTED 5-MANIFOLDS

Let X^{2n+1} denote a connected, oriented manifold of odd dimension. By definition, an *almost contact structure* on X is a rank $2n$ -distribution $\xi \subset TX$ together with a symplectic structure σ on the vector bundle $\xi \rightarrow X$. A *contact structure* is an almost contact structure such that the symplectic form σ on ξ is of the form $(d\alpha)|_\xi$, where α is a nowhere vanishing 1-form on X that defines ξ in the sense that the kernel distribution $\ker \alpha$ equals ξ . The 1-form α is called a *contact form*.

If (ξ, σ) is an almost contact structure, we can choose a complex structure on ξ compatible with the symplectic form σ and hence define Chern classes $c_k(\xi) \in H^{2k}(X)$. These classes do not depend on the choice of compatible complex structure, because the space of complex structures compatible with a given symplectic form is contractible. However, they depend on the choice of symplectic structure. For a contact structure we can choose complex structures compatible with the symplectic form $(d\alpha)|_\xi$ for a defining 1-form α . Since any two defining 1-forms only differ by multiplication with a nowhere zero function on X , it follows that the Chern classes $c_k(\xi)$ of a contact structure depend only on the contact distribution ξ , not on the choice of contact form α .

The first Chern class of an almost contact structure ξ is related to the second Stiefel-Whitney class of the manifold X in the following way:

Lemma 3. *Let ξ be an almost contact structure on X . Then $c_1(\xi) \equiv w_2(X) \bmod 2$.*

Proof. By the Whitney sum formula for $TX = \xi \oplus \mathbb{R}$,

$$w_2(X) = w_2(\xi) \cup w_0(\mathbb{R}) = w_2(\xi).$$

Since $\xi \rightarrow X$ is a complex vector bundle, with complex structure compatible with σ , we have $w_2(\xi) \equiv c_1(\xi) \bmod 2$. This implies the claim. \square

Suppose that ξ_t , for $t \in [0, 1]$, is a smooth family of contact structures on a closed manifold X . We can choose a smooth family of 1-forms α_t defining ξ_t . Using the Moser technique, one can prove that there exists a smooth family ψ_t of self-diffeomorphisms of X with $\psi_0 = Id_X$ such that $\psi^* \alpha_t = f_t \alpha_0$, for smooth functions f_t on X [16]. This implies the following theorem of J. W. Gray [8].

Theorem 4. *Let $\xi_t, t \in [0, 1]$, be a smooth family of contact structures on a closed manifold X . Then there exists an isotopy $\psi_t, t \in [0, 1]$, of diffeomorphisms of X such that $\psi_t^* \xi_t = \xi_0$.*

Because of this theorem, we call contact structures ξ, ξ' which can be deformed into each other by a smooth family of contact structures *isotopic*. We call almost contact structures *homotopic*, if they can be connected by a smooth family of almost contact structures. The contact structures in an isotopy class or the almost contact structures in a homotopy class all have the same Chern classes. We can also consider (almost) contact structures ξ, ξ' which are permuted by an orientation-preserving self-diffeomorphism ψ of X , in the sense that $\psi^* \xi' = \xi$.

Definition 5. We call almost contact structures and contact structures on an oriented manifold X *equivalent*, if they can be made identical by a combination of deformations (homotopies and isotopies, respectively) and by orientation-preserving self-diffeomorphisms of X .

The existence question for *almost contact structures* on 5-manifolds was settled by the following theorem of Gray [8].

Theorem 6. *Let X be a closed, orientable 5-manifold. Then X admits an almost contact structure if and only if $W_3(X) = 0$.*

Here $W_3(X) \in H^3(X)$ is the third integral Stiefel-Whitney class, defined as the image of $w_2(X)$ under the Bockstein homomorphism.

The existence of *contact structures* on simply-connected 5-manifolds was proved by H. Geiges [6]. He also proved a classification theorem for almost contact structures on simply-connected 5-manifolds up to homotopy:

Theorem 7. *Let X be a simply-connected, closed 5-manifold.*

- (a) *Every class in $H^2(X)$ that reduces mod 2 to $w_2(X)$ arises as the first Chern class of an almost contact structure. Two almost contact structures ξ_0, ξ_1 are homotopic if and only if $c_1(\xi_0) = c_1(\xi_1)$.*
- (b) *Every homotopy class of almost contact structures admits a contact structure.*

A different proof for the existence of contact structures on simply-connected 5-manifolds can be found in [13, 14]. The fact, that two almost contact structures are homotopic if they have the same first Chern class holds more generally for closed, oriented 5-manifolds without 2-torsion in $H^2(X)$. For a proof see [9, Theorem 8.18].

We want to prove the following theorem, which is a consequence of Barden's classification theorem.

Theorem 8. *Suppose that X is a simply-connected, closed, oriented 5-manifold. Let $c, c' \in H^2(X)$ be classes with the same divisibility and whose mod 2 reduction is $w_2(X)$. Then there exists an orientation preserving self-diffeomorphism $\phi: X \rightarrow X$ such that $\phi^* c' = c$.*

Here *divisibility* means the maximal divisibility as an element in the free abelian group $H^2(X)$. The divisibility is zero if and only if the class is zero itself. The proof of the theorem uses the following lemma.

Lemma 9. *Let G be a finitely generated free abelian group of rank n . Suppose $\alpha \in \text{Hom}(G, \mathbb{Z})$ is indivisible. Then there exists a basis e_1, \dots, e_n of G such that $\alpha(e_1) = 1$ and $\alpha(e_i) = 0$ for $i > 1$.*

Proof. The kernel of α is a free abelian subgroup of G of rank $n-1$. Let e_2, \dots, e_n be a basis of $\ker \alpha$. The image of α in \mathbb{Z} is a subgroup, hence of the form $m\mathbb{Z}$. Since α is indivisible, $m = 1$, so there exists an $e_1 \in G$ such that $\alpha(e_1) = 1$. The set e_1, \dots, e_n is linearly independent. They also span G , because if $g \in G$ is some element, then $\alpha(g - \alpha(g)e_1) = 0$, hence $g = \alpha(g)e_1 + \sum_{i \geq 2} \lambda_i e_i$. \square

We can now prove Theorem 8.

Proof. By the Universal Coefficient Theorem, $H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z})$ since X is simply-connected. Hence we can view c, c' as homomorphisms on $H_2(X)$ with values in \mathbb{Z} . Let $p: \mathbb{Z} \rightarrow \mathbb{Z}_2$ denote mod 2 reduction. The assumption on c and c' is equivalent to

$$w_2(X) = p \circ c = p \circ c',$$

as homomorphisms on $H_2(X)$ with values in \mathbb{Z}_2 . Since c and c' have the same divisibility, we can write

$$c = k\alpha, \quad c' = k\alpha'$$

with $\alpha, \alpha' \in \text{Hom}(H_2(X), \mathbb{Z})$ indivisible. Let $H_2(X) = G \oplus \text{Tor}H_2(X)$ with G free abelian. Since c and c' are homomorphism to \mathbb{Z} they vanish on $\text{Tor}H_2(X)$. By Lemma 9 there exist bases e_1, \dots, e_n and e'_1, \dots, e'_n of G such that

$$\alpha(e_1) = 1 = \alpha'(e'_1), \quad \alpha(e_k) = 0 = \alpha'(e'_k) \quad \forall k > 1.$$

Let θ be the group automorphism of $H_2(X)$ given by $\theta(e_k) = e'_k$ for all $k \geq 1$, and which is the identity on $\text{Tor}H_2(X)$. Then

$$(c' \circ \theta)(e_k) = c'(e'_k) = c(e_k) \quad \forall k \geq 1.$$

Hence $c' \circ \theta = c$ on the free abelian subgroup G . This equality holds on all of $H_2(X)$ because c and c' vanish on the torsion subgroup. By the assumption above, this implies that $w_2(X) \circ \theta = w_2(X)$. Moreover, since θ is the identity on $\text{Tor}H_2(X)$, it preserves the linking form. By Theorem 1, the automorphism θ is induced by an orientation preserving self-diffeomorphism $\phi: X \rightarrow X$ such that $\phi_* = \theta$. We have

$$c(\lambda) = c'(\phi_*\lambda) = (\phi^*c')(\lambda), \quad \text{for all } \lambda \in H_2(X).$$

Hence $\phi^*c' = c$. \square

We get the following corollary for almost contact structures.

Corollary 10. *Let X be a simply-connected, closed, oriented 5-manifold. Then two almost contact structures ξ_0 and ξ_1 on X are equivalent if and only if $c_1(\xi_0)$ and $c_1(\xi_1)$ have the same divisibility in integral cohomology.*

One direction is clear, because homotopies do not change the Chern class and self-diffeomorphisms of the manifold do not change the divisibility. The other direction follows from Theorem 8 and the first part of Theorem 7.

Definition 11. For an almost contact structure ξ on a simply-connected 5-manifold X , we denote the divisibility of $c_1(\xi)$ as a class in the free abelian group $H^2(X)$ by $d(\xi)$.

We call $d(\xi)$ the *level* of the almost contact structure ξ . By Corollary 10, almost contact structures and hence contact structures on a simply-connected 5-manifold X naturally form a "spectrum" consisting of levels which are indexed by the divisibility of the first Chern class. Two contact structures on X are equivalent as almost contact structures if and only if they lie on the same level. By Lemma 3, simply-connected spin 5-manifolds have only even levels and non-spin 5-manifolds only odd levels. In Section 7, we will use invariants from contact homology to investigate the "fine-structure" of contact structures on each level in this spectrum. For instance, O. van Koert [13] has shown that for many simply-connected 5-manifolds the lowest level, given by divisibility 0, contains infinitely many inequivalent contact structures.

4. TOPOLOGY OF CIRCLE BUNDLES

In this section we collect some results on the topology of circle bundles. In particular, we determine which simply-connected closed 5-manifolds can arise as circle bundles over 4-manifolds.

Let M be a closed, connected, oriented n -manifold. For a second integral cohomology class c on M consider the map

$$\langle c, - \rangle : H_2(M) \rightarrow \mathbb{Z},$$

given by evaluation.

Definition 12. We call the class c *indivisible* if $\langle c, - \rangle$ is surjective.

Clearly, if the class c is indivisible, then c cannot be written as $c = ka$, with $k > 1$ and $a \in H^2(M)$. By Poincaré duality it follows that a class $c \in H^2(M)$ is indivisible if and only if the map

$$c \cup : H^{n-2}(M) \rightarrow H^n(M) \cong \mathbb{Z}$$

is surjective.

Suppose that $\pi : X \rightarrow M$ is the total space of a circle bundle over M with Euler class $e \in H^2(M)$. For the following proofs we will need two results which are probably well known, but included here for completeness. The first result is related to the exact Gysin sequence [17]:

$$\dots \xrightarrow{\pi^*} H^k(X) \xrightarrow{\pi_*} H^{k-1}(M) \xrightarrow{\cup e} H^{k+1}(M) \xrightarrow{\pi^*} H^{k+1}(X) \xrightarrow{\pi_*} \dots$$

The homomorphism π_* is called integration along the fibre and can be characterized in the following way.

Lemma 13. *Integration along the fibre $\pi_*: H^{k+1}(X) \rightarrow H^k(M)$ is Poincaré dual to the map $\pi_*: H_{n-k}(X) \rightarrow H_{n-k}(M)$ induced by the projection.*

Proof. We only sketch the proof. Let $\pi: D \rightarrow M$ denote the disc bundle with Euler class e . Then $X \cong \partial D$ and integration along the fibre

$$\pi_*: H^{k+1}(\partial D) \rightarrow H^k(M)$$

is given by (see [17])

$$H^{k+1}(\partial D) \xrightarrow{\delta} H^{k+2}(D, \partial D) \xrightarrow{\tau^{-1}} H^k(D) \xrightarrow{\pi^*} H^k(M).$$

Here δ denotes the connecting homomorphism in the long exact sequence of the pair $(D, \partial D)$ and τ^{-1} the inverse of the Thom isomorphism. The proof follows from this. \square

The second result is related to the long exact homotopy sequence associated to the fibration

$$\dots \rightarrow \pi_2(M) \xrightarrow{\partial} \pi_1(S^1) \rightarrow \pi_1(X) \xrightarrow{\pi_*} \pi_1(M) \rightarrow 1.$$

Lemma 14. *The map $\partial: \pi_2(M) \rightarrow \pi_1(S^1) \cong \mathbb{Z}$ in the long exact homotopy sequence for fibre bundles is given by*

$$\pi_2(M) \xrightarrow{h} H_2(M) \xrightarrow{\langle e, - \rangle} \mathbb{Z}$$

where h denotes the Hurewicz homomorphism.

Proof. Let $f: S^2 \rightarrow M$ be a continuous map and $E = f^*X$ the pull-back S^1 -bundle over S^2 . By naturality of the long exact homotopy sequence there is a commutative diagram

$$\begin{array}{ccccccccc} \pi_2(X) & \longrightarrow & \pi_2(M) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(M) \\ \uparrow & & f_* \uparrow & & = \uparrow & & \uparrow & & \uparrow \\ \pi_2(E) & \longrightarrow & \pi_2(S^2) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(E) & \longrightarrow & 1 \end{array}$$

Since f can represent any element in $\pi_2(M)$ and the equation $f^*(e(X)) = e(E)$ holds by naturality of the Euler class it suffices to prove the claim for M equal to S^2 . We then have to prove that the map $\partial: \pi_2(S^2) \rightarrow \pi_1(S^1)$ is multiplication $\mathbb{Z} \xrightarrow{a} \mathbb{Z}$ by the Euler number $a = \langle e(E), [S^2] \rangle$.

By the exact sequence above it follows that $\pi_1(S^1) = \mathbb{Z}$ maps surjectively onto $\pi_1(E)$. Hence $\pi_1(E)$ is an abelian group. Therefore we have to prove that $H^2(E) \cong H_1(E) \cong \pi_1(E)$ is equal to $\mathbb{Z}/a\mathbb{Z}$. This follows from the following part of the Gysin sequence:

$$H^0(S^2) \xrightarrow{\cup e} H^2(S^2) \rightarrow H^2(E) \xrightarrow{\pi_*} H^1(S^2) = 0.$$

\square

Lemma 15. *The Euler class e is indivisible if and only if $\pi_*: H_1(X) \rightarrow H_1(M)$ is an isomorphism. Both statements are equivalent to the fibre $S^1 \subset X$ being null-homologous.*

Proof. Consider the following part of the Gysin sequence:

$$\dots \longrightarrow H^{n-2}(M) \xrightarrow{\cup e} H^n(M) \longrightarrow H^n(X) \xrightarrow{\pi_*} H^{n-1}(M) \longrightarrow 0.$$

This shows that e is indivisible if and only if $\pi_*: H^n(X) \rightarrow H^{n-1}(M)$ is an isomorphism, in other words

$$\pi_*: H_1(X) \longrightarrow H_1(M)$$

is an isomorphism. The long exact homotopy sequence of the fibration $S^1 \rightarrow X \rightarrow M$ induces by abelianization an exact sequence

$$H_1(S^1) \longrightarrow H_1(X) \longrightarrow H_1(M) \longrightarrow 0.$$

Hence we see that e is indivisible if and only if the fibre $S^1 \subset X$ is null-homologous. \square

From the long exact homotopy sequence above, we see that the fibre is *null-homotopic* if and only if $\partial: \pi_2(M) \rightarrow \pi_1(S^1)$ is surjective. By Lemma 14, this happens if and only if $\langle e, - \rangle$ is surjective on spherical classes. Both statements are equivalent to

$$\pi_*: \pi_1(X) \rightarrow \pi_1(M)$$

being an isomorphism.

Lemma 16. *X is simply-connected if and only if M is simply-connected and e is indivisible.*

Proof. If X is simply-connected, the long exact homotopy sequence shows that $\pi_1(M) = 1$ and $\partial: \pi_2(M) \rightarrow \pi_1(S^1)$ is surjective. Hence M is simply-connected and the surjectivity of ∂ implies that e is indivisible. Conversely, suppose that M is simply-connected and e is indivisible. Then the Hurewicz map $h: \pi_2(M) \rightarrow H_2(M)$ is an isomorphism and it follows that ∂ is surjective. The long exact homotopy sequence then implies the exact sequence $1 \rightarrow \pi_1(X) \rightarrow 1$. Hence $\pi_1(X) = 1$. \square

Lemma 17. *Suppose the first Betti number of M vanishes, $b_1(M) = 0$. Then the map $\pi^*: H^2(M) \rightarrow H^2(X)$ is surjective with kernel $\mathbb{Z} \cdot e$.*

Proof. We consider the following part of the Gysin sequence:

$$H^0(M) \xrightarrow{\cup e} H^2(M) \xrightarrow{\pi^*} H^2(X) \longrightarrow H^1(M).$$

By assumption, $H^1(M) = 0$. Hence $\pi^*: H^2(M) \rightarrow H^2(X)$ is surjective with kernel $H^0(M) \cup e = \mathbb{Z} \cdot e$. \square

Lemma 18. *The image of $\pi_*: H_2(X) \rightarrow H_2(M)$ is the kernel of $\langle e, - \rangle$.*

Proof. We consider the following part of the Gysin sequence:

$$H^{n-1}(X) \xrightarrow{\pi_*} H^{n-2}(M) \xrightarrow{\cup e} H^n(M) \cong \mathbb{Z}.$$

A class $\alpha \in H^{n-2}(M)$ is in the image of π_* if and only if $e \cup \alpha = 0$, which is the case if and only if the Poincaré dual $b = PD(\alpha) \in H_2(M)$ satisfies $\langle e, b \rangle = 0$. Since integration along the fibre

$$\pi_*: H^{n-1}(X) \rightarrow H^{n-2}(M)$$

is by Lemma 13 Poincaré dual to

$$\pi_*: H_2(X) \rightarrow H_2(M),$$

this proves the claim. \square

We now determine when the total space X is spin.

Lemma 19. *The total space X is spin if and only if $w_2(M) \equiv \alpha e \pmod{2}$ for some $\alpha \in \{0, 1\}$, i.e. if and only if M is spin or $w_2(M) \equiv e \pmod{2}$.*

Proof. We claim that the following relation holds:

$$w_2(X) = \pi^* w_2(M).$$

This follows because the tangent bundle of X is given by $TX = \pi^* TM \oplus \mathbb{R}$ and the Whitney sum formula implies $w_2(TX) = w_2(\pi^* TM) \cup w_0(\mathbb{R}) = \pi^* w_2(TM)$. Hence X is spin if and only if $w_2(M)$ is in the kernel of π^* .

We consider the following part of the \mathbb{Z}_2 -Gysin sequence:

$$H^0(M; \mathbb{Z}_2) \xrightarrow{\cup \bar{e}} H^2(M; \mathbb{Z}_2) \xrightarrow{\pi^*} H^2(X; \mathbb{Z}_2),$$

where \bar{e} denotes the mod 2 reduction of e . Hence the kernel of π^* is $\{0, \bar{e}\}$. This implies the claim. \square

We now specialize to the case where the dimension of M is equal to 4.

Theorem 20. *Let X be a simply-connected closed oriented 5-manifold which is a circle bundle over a closed oriented 4-manifold M . Then M is simply-connected and the Euler class e is indivisible. Moreover, the integral homology and cohomology of X are torsion free and given by:*

- $H_0(X) \cong H_5(X) \cong \mathbb{Z}$
- $H_1(X) \cong H_4(X) \cong 0$
- $H_2(X) \cong H_3(X) \cong \mathbb{Z}^{b_2(M)-1}$.

Proof. We only have to prove that the cohomology of X is torsion free and the formula for $H_2(X)$. The cohomology groups $H^0(X)$, $H^1(X)$ and $H^5(X)$ are always torsion free for an oriented 5-manifold X . We have the following part of the Gysin sequence:

$$\dots \rightarrow H^3(M) \xrightarrow{\pi^*} H^3(X) \xrightarrow{\pi_*} H^2(M) \rightarrow \dots$$

By assumption, $H^3(M) = 0$. Therefore the homomorphism π_* injects $H^3(X)$ into $H^2(M)$, which is torsion free by the assumption that M is simply-connected. Hence $H^3(X)$ is torsion free itself. It remains to consider $H^2(X)$ and $H^4(X)$. By the Universal Coefficient Theorem and Poincaré duality, $H^2(X)$ is torsion free if and only if $H_1(X)$ is torsion free, if and only if $H^4(X)$ is torsion free. Since $H_1(X) = 0$, we see that $H^2(X)$ and $H^4(X)$ are torsion free.

By Lemma 17 we have $H^2(X) \cong H^2(M)/\mathbb{Z} \cdot e$. Since $H^2(M)$ is torsion free and e is indivisible, $H^2(M)/\mathbb{Z} \cdot e \cong \mathbb{Z}^{b_2(M)-1}$. This implies the formula for $H_2(X) \cong H_3(X)$. \square

With Proposition 2, we get the following corollary (this has also been proved in [4]).

Corollary 21. *Let M be a simply-connected closed oriented 4-manifold and X the circle bundle over M with indivisible Euler class e . Then X is diffeomorphic to*

- (a) $X = \#(b_2(M) - 1)S^2 \times S^3$ if X is spin
- (b) $X = \#(b_2(M) - 2)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ if X is not spin.

The first case occurs if and only if M is spin or $w_2(M) \equiv e \pmod{2}$.

Since every closed oriented 4-manifold is Spin^c and hence $w_2(M)$ is the mod 2 reduction of an integral class, it follows as a corollary that every closed simply-connected 4-manifold M is diffeomorphic to the quotient of a free and smooth S^1 -action on $\#(b_2(M) - 1)S^2 \times S^3$.

5. THE BOOTHBY-WANG CONSTRUCTION

We want to construct circle bundles over symplectic manifolds M whose Euler class is represented by the symplectic form. Since the Euler class is an element of the integral cohomology group $H^2(M)$, the symplectic form has to represent an integral cohomology class in $H^2(M; \mathbb{R})$, i.e. it has to lie in the image of the natural homomorphism

$$H^2(M) \rightarrow H^2(M; \mathbb{R}) \xrightarrow{\cong} H_{DR}^2(M).$$

The existence of such a symplectic form is guaranteed by the following argument (this argument is from [7, Observation 4.3]): Let (M, ω) be a closed symplectic manifold. For every Riemannian metric on M , there exists a small ϵ -ball B_ϵ around 0 in the space of harmonic 2-forms on M such that every element in $\omega + B_\epsilon$ is symplectic. Since the set of classes in $H^2(M; \mathbb{R})$ represented by these elements is open, there exists a symplectic form which represents a rational cohomology class. By multiplication with a suitable integer, we can find a symplectic form which represents an integral class. If we want, we can choose the integer such that the class is indivisible. Note also that all symplectic forms in $\omega + B_\epsilon$ can be connected to ω by a smooth path of symplectic forms. This implies that they all have the same canonical class K as ω .

We fix the following data:

- (a) A closed connected symplectic manifold (M^{2n}, ω) with symplectic form ω , representing an integral cohomology class in $H^2(M; \mathbb{R})$.
- (b) An integral lift $[\omega]_{\mathbb{Z}} \in H^2(M)$ of $[\omega] \in H_{DR}^2(M)$.

Let $\pi: X \rightarrow M$ be the principal circle bundle over M with Euler class $e(X) = [\omega]_{\mathbb{Z}}$. By a theorem of Kobayashi [12] we can choose a $U(1)$ -connection A on $X \rightarrow M$ whose curvature form F is $\frac{2\pi}{i}\omega$. Then A is a 1-form on X with values

in $\mathfrak{u}(1) \cong i\mathbb{R}$ which is invariant under the S^1 -action and there are the following relations, coming from the definition of a connection on a principal bundle:

$$\begin{aligned} dA &= \pi^* F \\ A(R) &= 2\pi i \end{aligned}$$

Here R denotes the fundamental vector field generated by the action of the element $2\pi i \in \mathfrak{u}(1)$. An orbit of R , topologically a fibre of X , has period 1.

Proposition 22. *Define the real valued 1-form $\lambda = \frac{1}{2\pi i} A$ on X . Then λ is a contact form on X with*

$$\begin{aligned} d\lambda &= -\pi^* \omega \\ \lambda(R) &= 1. \end{aligned}$$

Proof. We have the relations

$$\begin{aligned} dA &= -2\pi i \pi^* \omega \\ A(R) &= 2\pi i. \end{aligned}$$

This implies the corresponding relations for λ . The tangent bundle of X splits as $TX \cong \mathbb{R} \oplus \pi^* TM$, where the trivial \mathbb{R} -summand is spanned by the vector field R . Fix a point of X and choose a basis (R, v_1, \dots, v_{2n}) of its tangent space, where the v_i form an oriented basis of the kernel of λ . Then

$$\begin{aligned} \lambda \wedge (d\lambda)^n(R, v_1, \dots, v_{2n}) &= (d\lambda)^n(v_1, \dots, v_{2n}) \\ &= (-1)^n \omega^n(\pi_* v_1, \dots, \pi_* v_{2n}) \\ &\neq 0. \end{aligned}$$

Hence $\lambda \wedge (d\lambda)^n$ is a volume form on X , and λ is contact. \square

Remark 23. If we define the orientation on X via the splitting $TX \cong \mathbb{R} \oplus \pi^* TM$, where the trivial \mathbb{R} -summand is oriented by R and TM by ω , then λ is a positive contact form if n is even and negative otherwise.

Definition 24. The contact structure ξ on the closed oriented manifold X^{2n+1} , defined by the contact form λ above, is called the *Boothby-Wang contact structure* associated to the symplectic manifold (M, ω) . Since $d\lambda(R) = 0$, the Reeb vector field of λ is given by the vector field R along the fibres.

For the original construction see [2].

Proposition 25. *If λ' is another contact form, defined by a different connection A' as above, then the associated contact structure ξ' is isotopic to ξ .*

Proof. The connection A' is an S^1 -invariant 1-form on X with

$$\begin{aligned} dA' &= dA \\ A'(R) &= A(R). \end{aligned}$$

Hence $A' - A = \pi^* \alpha$ for some closed 1-form α on M . Define $A_t = A + \pi^* t \alpha$ for $t \in \mathbb{R}$. Then A_t is a connection on X with curvature $-2\pi i \omega$ for all t . Let $\lambda_t = \lambda + \pi^* (\frac{1}{2\pi i} t \alpha)$. Then λ_t is a contact form on X for all $t \in [0, 1]$ with $\lambda_0 = \lambda$ and $\lambda_1 = \lambda'$. Therefore, ξ and ξ' are isotopic through the contact structures defined by λ_t . \square

The Chern classes of ξ are given by the Chern classes associated to ω in the following way.

Lemma 26. *Let $X \rightarrow M$ be a Boothby-Wang fibration with contact structure ξ . Then $c_i(\xi) = \pi^* c_i(TM, \omega)$ for all $i \geq 0$. The manifold X is spin, if and only if*

$$c_1(M) \equiv \alpha[\omega]_{\mathbb{Z}} \text{ mod } 2,$$

for some $\alpha \in \{0, 1\}$.

Proof. Let J be a compatible almost complex structure for ω on M . Then there exists a compatible complex structure J' for ξ on X such that $\pi^*(TM, J) \cong (\xi, J')$ as complex vector bundles. The naturality of characteristic classes proves the first claim. The second claim follows from Lemma 19 and $c_1(M) \equiv w_2(M) \text{ mod } 2$. \square

6. THE CONSTRUCTION FOR SYMPLECTIC 4-MANIFOLDS

We fix the following data:

- (a) A closed, simply-connected, symplectic 4-manifold (M, ω) with symplectic form ω representing an integral cohomology class in $H^2(M; \mathbb{R})$, given by the argument at the beginning of Section 5. Since $H^2(M)$ is torsion free, $[\omega]$ has a unique integral lift, denoted by $[\omega]_{\mathbb{Z}} \in H^2(M)$. We sometimes denote the integral lift also by $[\omega]$ or ω . We assume that $[\omega]_{\mathbb{Z}}$ is indivisible.
- (b) Let $\pi: X \rightarrow M$ be the principal S^1 -bundle over M with Euler class $e(X) = [\omega]_{\mathbb{Z}}$. Then X is a closed, simply-connected, oriented 5-manifold with torsion-free homology by Theorem 20.
- (c) Let λ be a Boothby-Wang contact form on X with associated contact structure ξ . By Proposition 25, the contact structure ξ does not depend on λ up to isotopy.

By Corollary 21, the 5-manifold X is diffeomorphic to

- $\#(b_2(M) - 1)S^2 \times S^3$ if X is spin
- $\#(b_2(M) - 2)S^2 \times S^3 \# S^2 \times S^3$ if X is not spin.

Hence the same abstract, closed, simply-connected 5-manifold X with torsion free homology can be realized in several different ways as a Boothby-Wang fibration over different simply-connected symplectic 4-manifolds M and therefore admits many, possibly non-equivalent, contact structures.

Definition 27. Let $d(\xi) \geq 0$ denote the divisibility of $c_1(\xi)$ in the free abelian group $H^2(X)$. Similarly, we denote by $d(K)$ the divisibility of the canonical class $K = -c_1(M) \in H^2(M)$ of the symplectic structure ω .

Note that X is spin if and only if $d(\xi)$ is even by Lemma 3. With Corollary 10 we get:

Proposition 28. *Suppose that (M', ω') is another closed, simply-connected, symplectic 4-manifold with integral and indivisible symplectic form ω' . Denote the associated Boothby-Wang total space by (X', ξ') .*

- (a) *The simply-connected 5-manifolds X and X' are diffeomorphic if and only if $b_2(M) = b_2(M')$ and $d(\xi) \equiv d(\xi') \pmod{2}$.*
- (b) *If X and X' are diffeomorphic and $d(\xi) = d(\xi')$, then ξ and ξ' are equivalent as almost contact structures.*

The divisibility $d(\xi)$ can be calculated in the following way: By Lemma 17, the bundle projection π defines an isomorphism

$$\pi^*: H^2(M)/\mathbb{Z}\omega \xrightarrow{\cong} H^2(X),$$

and by Lemma 26 we have

$$\pi^*c_1(M) = c_1(\xi).$$

Let $[c_1(M)]$ denote the image of $c_1(M)$ in the quotient $H^2(M)/\mathbb{Z}\omega$, which is free abelian since ω is indivisible. We will use π^* to identify

$$\begin{aligned} H^2(X) &= H^2(M)/\mathbb{Z}\omega, \text{ and} \\ c_1(\xi) &= [c_1(M)]. \end{aligned}$$

Then $d(\xi)$ is also the divisibility of the class $[c_1(M)]$. If the second Betti number of M is equal to 1, then $H^2(X) = 0$ and $d(\xi) = 0$ trivially. For $b_2(M) > 1$ we have:

Lemma 29. *The divisibility $d(\xi)$ is the maximal integer d such that*

$$c_1(M) = dR + \gamma\omega$$

where γ is some integer and $R \in H^2(M)$ not a multiple of ω .

An important fact is the following:

Lemma 30. *The integer $d(\xi)$ is always a multiple of $d(K)$.*

Proof. We can write $c_1(M) = d(K)W$ where W is a class in $H^2(M)$. Then $[c_1(M)] = d(K)[W]$ in $H^2(M)/\mathbb{Z}\omega$. Since $d(\xi)$ is the divisibility of $[c_1(M)]$, the integer $d(\xi)$ has to be a multiple of $d(K)$. \square

Hence the possible levels of Boothby-Wang contact structures are restricted to the multiples of the divisibility of the canonical class.

7. CONTACT HOMOLOGY

In this section we consider invariants derived from contact homology. We only take into account the *classical contact homology* $H_*^{cont}(X, \xi)$ which is a graded supercommutative algebra, defined using rational holomorphic curves with one positive puncture and several negative punctures in the symplectization of the contact

manifold. We use a variant of this theory for the so-called Morse-Bott case, described in [3] and in Section 2.9.2. in [5].

We are going to associate to each Boothby-Wang fibration $\pi: X \rightarrow M$ as in the previous section a graded commutative algebra $\mathfrak{A}(X, M)$. Choose a basis B_1, \dots, B_N of $H_2(X)$, where $N = b_2(X) = b_2(M) - 1$ and let

$$A_n = \pi_* B_n \in H_2(M), \quad 1 \leq n \leq N.$$

Note that

$$c_1(B_n) := \langle c_1(\xi), B_n \rangle = \langle c_1(M), A_n \rangle = c_1(A_n).$$

Choose a class $A_0 \in H_2(M)$ such that

$$\omega(A_0) = 1.$$

This is possible, because ω was assumed indivisible. The classes A_0, A_1, \dots, A_N form a basis of $H_2(M)$. We consider variables

$$z = (z_1, \dots, z_N), \text{ and} \\ q = \{q_{k,i}\}_{k \in \mathbb{N}, 0 \leq i \leq a},$$

where $a = b_2(M) + 1$ and \mathbb{N} denotes the set of positive integers. They have degrees defined by

$$\deg(z_n) = -2c_1(B_n) \\ \deg(q_{k,i}) = \deg \Delta_i - 2 + 2c_1(A_0)k,$$

where $\deg \Delta_i$ is given by

$$\deg \Delta_i = \begin{cases} 0 & \text{if } i = 0 \\ 2 & \text{if } i = 1, \dots, b_2(M) \\ 4 & \text{if } i = b_2(M) + 1. \end{cases}$$

In our situation the degree of all variables is even (hence the algebra we are going to define is truly commutative, not only supercommutative).

Definition 31. We define the following algebras.

- $\mathfrak{L}(X) = \mathbb{C}[H_2(X; \mathbb{Z})]$ is the graded commutative ring of Laurent polynomials in the variables z with coefficients in \mathbb{C} .
- $\mathfrak{A}(X, M) = \bigoplus_{d \in \mathbb{Z}} \mathfrak{A}_d(X, M)$ is the graded commutative algebra of polynomials in the variables q with coefficients in $\mathfrak{L}(X)$.

A homomorphism ϕ of graded commutative algebras $\mathfrak{A}, \mathfrak{A}'$ over $\mathfrak{L}(X)$

$$\phi: \mathfrak{A} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{A}_d \rightarrow \mathfrak{A}' = \bigoplus_{d \in \mathbb{Z}} \mathfrak{A}'_d$$

is a homomorphism of rings, which is the identity on $\mathfrak{L}(X)$ and such that $\phi(\mathfrak{A}_d) \subset \mathfrak{A}'_d$ for all $d \in \mathbb{Z}$.

Lemma 32. *The following statements hold:*

- Up to isomorphism, the ring $\mathfrak{L}(X)$ does not depend on the choice of basis B_1, \dots, B_N for $H_2(X)$.

- (b) For fixed $\mathfrak{L}(X)$, the algebra $\mathfrak{A}(X, M)$ does not depend, up to isomorphism over $\mathfrak{L}(X)$, on the choice of the class $A_0 \in H_2(M)$ as above.

Proof. Let $\overline{B_1}, \dots, \overline{B_N}$ be another basis of $H_2(X)$ and $\overline{\mathfrak{L}(X)}$ the associated ring, generated by variables \overline{z} . Then there exists matrix

$$(\beta_{mn}) \in SL(N, \mathbb{Z})$$

with $1 \leq m, n \leq N$, such that

$$\overline{B_m} = \sum_{n=1}^N \beta_{mn} B_n.$$

Define a homomorphism $\phi: \overline{\mathfrak{L}(X)} \rightarrow \mathfrak{L}(X)$ via

$$\overline{z_m} \mapsto \prod_{n=1}^N z_n^{\beta_{mn}},$$

for all $1 \leq m \leq N$. Then ϕ preserves degrees and is an isomorphism, since the matrix (β_{mn}) is invertible.

Let $\overline{A_0}$ be another element in $H_2(M)$ such that $\omega(\overline{A_0}) = 1$ and $\overline{\mathfrak{A}(X, M)}$ the associated algebra, generated by variables \overline{q} . Then there exists a vector

$$(\alpha_n) \in \mathbb{Z}^N,$$

with $1 \leq n \leq N$, such that

$$\overline{A_0} = A_0 + \sum_{n=1}^N \alpha_n A_n.$$

Define a homomorphism $\psi: \overline{\mathfrak{A}(X, M)} \rightarrow \mathfrak{A}(X, M)$ via

$$\overline{q_{k,i}} \mapsto q_{k,i} \prod_{n=1}^N z_n^{-k\alpha_n}, \quad k \in \mathbb{N}, 0 \leq i \leq a,$$

and which is the identity on $\mathfrak{L}(X)$. Then ψ preserves degrees and is invertible. \square

We choose a class $A_0 \in H_2(M)$ with $\omega(A_0) = 1$ and denote $c_1(A_0)$ by Δ . Hence the degrees of the variables $q_{k,i}$ are equal to

$$\deg(q_{k,i}) = \deg \Delta_i - 2 + 2\Delta k.$$

The integer Δ has the following properties.

Lemma 33. *The following relations hold:*

- (a) Let $c_1(M) = d(\xi)R + \gamma\omega$ for some class $R \in H^2(M)$ and integer $\gamma \in \mathbb{Z}$ as in Lemma 29. Then $\Delta \equiv \gamma \pmod{d(\xi)}$.
- (b) The greatest common divisor $\gcd(\Delta, d(\xi))$ is equal to $d(K)$. In particular, if $d(\xi) = 0$, then $\Delta = d(K)$.

Proof. Part (a) follows if we evaluate both sides on A_0 . To prove part (b), the integer $d(K)$ divides $d(\xi)$ by Lemma 30 and it divides $c_1(M)$, hence also Δ . On the other hand, there exists a homology class $B \in H_2(M)$ such that $d(K) = c_1(M)(B)$. By part (a)

$$d(K) = d(\xi)R(B) + \gamma\omega(B)$$

and $\gamma \equiv \Delta \pmod{d(\xi)}$. Hence there exist integers $x, y \in \mathbb{Z}$ such that $d(K) = xd(\xi) + y\Delta$. This proves the claim. \square

We are interested in the algebra $\mathfrak{A}(X, M)$ because of the following result, described in [5], Proposition 2.9.1:

Theorem 34. *For a Boothby-Wang fibration $X \rightarrow M$ as above, the Morse-Bott contact homology $H_*^{cont}(X, \xi)$ specialized¹ at $t = 0$ is isomorphic to $\mathfrak{A}(X, M)$.*

If two Boothby-Wang contact structures ξ and ξ' on X are equivalent, then their contact homologies are isomorphic. We now make the following assumptions:

- (a) The simply-connected 5-manifold X can be realized as the Boothby-Wang total space over another closed, simply-connected, symplectic 4-manifold (M', ω') where ω' represents an integral and indivisible class. This implies in particular that $b_2(M') = b_2(M)$ and both are equal to $a - 1$. Denote the canonical class of (M', ω') by K' and its divisibility by $d(K')$
- (b) We assume that ξ and ξ' are contact structures on the same level, hence $d(\xi') = d(\xi) =: d$ and both are equivalent as almost contact structures.
- (c) Let $\mathfrak{A}(X, M')$ denote the associated algebra over $\mathfrak{L}(X)$, generated by variables $\{q'_{l,j}\}$, with $l \in \mathbb{N}, 0 \leq j \leq a$. We set $\Delta' = c_1(A'_0)$ for a class A'_0 with $\omega'(A'_0) = 1$.

The following is the main theorem in this section:

Theorem 35. *The algebras $\mathfrak{A}(X, M)$ and $\mathfrak{A}(X, M')$ are isomorphic over $\mathfrak{L}(X)$ if and only if one of the following three conditions is satisfied:*

- $d \geq 1$ and both $d(K), d(K') \leq 3$
- $d = 0$ and $d(K) = d(K')$
- $d \geq 4$ and $d(K) = d(K') \geq 4$.

This shows that the isomorphism type of the contact homology for Boothby-Wang contact structures on the same level is strongly related to the divisibility of the canonical class of the symplectic structure.

The proof of this theorem is done in several steps. Let d denote the integer $d(\xi)$.

Definition 36. Suppose that $d \geq 1$. For $0 \leq b < d$ denote by Q_b the set of generators $\{q_{k,i}\}$ with

$$\deg(q_{k,i}) \equiv 2b \pmod{2d}.$$

Similarly denote by Q'_b the set of generators $\{q'_{l,j}\}$ with

$$\deg(q'_{l,j}) \equiv 2b \pmod{2d}.$$

¹Note that contact homology is actually a family of algebras which can be specialized at any $t \in H^*(X; \mathbb{R})$.

Remark 37. If $c_1(\xi) \neq 0$, the variables z_1, \dots, z_n which generate the ring $\mathfrak{L}(X)$ do not all have degree zero. Hence $\mathfrak{B}(X, M) = \mathfrak{A}(X, M)/\mathfrak{L}(X)$, which is an algebra over \mathbb{C} , does not inherit a natural grading in this case. However, since the degrees of the variables z_n are all multiples of $2d$, the algebra $\mathfrak{B}(X, M)$ has a grading by elements in \mathbb{Z}_{2d} . The images of the generators $q_{k,i}$ form generators for this infinite polynomial algebra and Q_b is the set of generators of degree $2b \bmod 2d$.²

The following lemma shows that there is a relation between the cardinality of the set Q_b of generators and the divisibility of the canonical class of the symplectic structure.

Lemma 38. *Assume that $d \geq 1$. Then the set Q_b is infinite if $d(K)$ divides one of the integers $b - 1, b, b + 1$ and empty otherwise.*

Proof. Suppose $d(K) = \gcd(\Delta, d)$ divides one of the integers $b + \epsilon$, with $\epsilon \in \{-1, 0, 1\}$. Then the equation

$$b = -\epsilon + \Delta k + d\alpha$$

has infinitely many solutions $k \geq 1$ with $\alpha \in \mathbb{Z}$. Choose an integer $0 \leq i \leq a$ with $\deg \Delta_i - 2 = -2\epsilon$. Then

$$\deg(q_{k,i}) = -2\epsilon + 2\Delta k \equiv 2b \bmod 2d$$

for infinitely many $k \geq 1$. Hence these $q_{k,i}$ are all in Q_b .

Conversely, suppose that $d(K)$ does not divide any of the integers $b + \epsilon$, with $\epsilon \in \{-1, 0, 1\}$. Suppose that Q_b contains an element $q_{l,j}$. We have $\deg(q_{l,j}) = -2\epsilon + 2\Delta l$ for some $\epsilon \in \{-1, 0, 1\}$. By assumption,

$$\deg(q_{l,j}) = -2\epsilon + 2\Delta l = 2b - 2d\alpha,$$

for some $\alpha \in \mathbb{Z}$. This implies

$$b + \epsilon = \Delta l + d\alpha.$$

This is impossible, since $d(K)$ divides the right side, but not the left side. \square

Example 39. Suppose that $d \geq 1$. If $d(K) \in \{1, 2, 3\}$, then Lemma 38 implies that Q_b is infinite for all $b = 0, \dots, d - 1$. If $d(K) \geq 4$ (and hence $d \geq 4$ as well), then at least one of the Q_b is empty, e.g. Q_2 is always empty in this case.

Lemma 38 implies the following relation between the cardinalities of the set of generators Q_b and Q'_b and the divisibilities of the canonical classes of the symplectic 4-manifolds M and M' .

Lemma 40. *Assume that $d \geq 4$ and at least one of the numbers $d(K), d(K')$ is ≥ 4 . Then the following two statements are equivalent:*

- (a) *There exists an integer $0 \leq b < d$ such that Q_b and Q'_b do not have the same cardinality (i.e. one of them is empty and the other infinite).*
- (b) *$d(K) \neq d(K')$.*

²This interpretation is due to K. Cieliebak.

Proof. Suppose that $d(K) = d(K')$. By Lemma 38, the sets Q_b and Q'_b have the same cardinality for all $0 \leq b < d$. Conversely, suppose that $d(K) \neq d(K')$; without loss of generality $d(K) < d(K')$. If $d(K) \in \{1, 2, 3\}$ let $b = 2$. Then Q_2 is infinite, while Q'_2 is empty (since $d(K') \geq 4$ by assumption). If $d(K) \geq 4$ let $b = d(K) - 1 \geq 3$. Then $d(K)$ divides $b + 1$, but $d(K')$ does not divide any of the integers $b - 1, b, b + 1$. Hence Q_b is infinite and Q'_b empty. \square

Using Lemma 40, we can prove the following.

Lemma 41. *Suppose that either (i) $d = 0$ or (ii) $d > 0$ and at least one of the numbers $d(K), d(K')$ is ≥ 4 . If the \mathbb{Z}_{2d} -graded polynomial algebras $\mathfrak{B}(X, M)$ and $\mathfrak{B}(X, M')$ over \mathbb{C} are isomorphic, then $d(K) = d(K')$.*

This implies one direction of Theorem 35, because an isomorphism of \mathfrak{A} -algebras induces an isomorphism of \mathfrak{B} -algebras.

Proof. Suppose that $d = 0$ and that there exists an isomorphism $\phi: \mathfrak{B}(X, M) \rightarrow \mathfrak{B}(X, M')$. In this case, both algebras are graded by the integers and the elements of lowest degree in $\mathfrak{B}(X, M)$ and $\mathfrak{B}(X, M')$ have degree $-2 + 2\Delta$ and $-2 + 2\Delta'$, respectively. Since ϕ has to preserve degree, this implies $\Delta = \Delta'$ and hence

$$d(K) = \gcd(\Delta, 0) = \Delta = \Delta' = \gcd(\Delta', 0) = d(K').$$

Now assume that $d > 0$ and at least one of $d(K), d(K')$ is ≥ 4 . By Lemma 30, the integer d is at least 4. Suppose that $d(K) \neq d(K')$ and there exists an isomorphism $\phi: \mathfrak{B}(X, M) \rightarrow \mathfrak{B}(X, M')$. Both algebras are freely generated by the images of the elements $\{q_{k,i}\}$ and $\{q'_{l,j}\}$, which we still denote by the same symbols.

By Lemma 40, there exists an integer $0 \leq b < d$ such that Q_b and Q'_b have different cardinality. Without loss of generality, we may assume that Q_b is empty and Q'_b infinite (otherwise we consider ϕ^{-1}). Let $q'_{r,s}$ be a generator in Q'_b . Then $q'_{r,s}$ is a polynomial in the images

$$\{\phi(q_{k,i})\}_{k \in \mathbb{N}, 0 \leq i \leq a},$$

with coefficients in \mathbb{C} and we can write

$$q'_{r,s} = f(\phi(q_{k_1,i_1}), \dots, \phi(q_{k_v,i_v})) \in \mathbb{C}[\phi(q_{k_1,i_1}), \dots, \phi(q_{k_v,i_v})].$$

The images $\phi(q_{k,i})$ are themselves polynomials in the variables $\{q'_{l,j}\}$ with coefficients in \mathbb{C} . Expressed as a polynomial in the variables $\{q'_{l,j}\}$, at least one of the images $\phi(q_{k_w,i_w})$, $1 \leq w \leq v$, must contain a summand of the form $\alpha q'_{r,s}$ with $\alpha \in \mathbb{C}$ non-zero. Since ϕ preserves degrees modulo 2, the element $\phi(q_{k_w,i_w})$ is homogeneous of degree

$$\deg(\phi(q_{k_w,i_w})) = \deg(\alpha q'_{r,s}) = \deg(q'_{r,s}) \equiv 2b \pmod{2d}.$$

This implies $\deg(q_{k_w,i_w}) \equiv 2b \pmod{2d}$, hence $q_{k_w,i_w} \in Q_b$. This is impossible, since $Q_b = \emptyset$. \square

The other direction of Theorem 35 follows from the next lemma.

Lemma 42. *Suppose that either (i) $d(K) = d(K')$ or (ii) both numbers $d(K), d(K')$ are ≤ 3 and $d \neq 0$. Then the algebras $\mathfrak{A}(X, M)$ and $\mathfrak{A}(X, M')$ are isomorphic over $\mathfrak{L}(X)$.*

Proof. We can choose a basis B_1, \dots, B_N of $H_2(X)$ such that

$$\begin{aligned} c_1(B_1) &= d(\xi) = d \\ c_1(B_n) &= 0, \quad \text{for all } 2 \leq n \leq N. \end{aligned}$$

Choose elements $A_0 \in H_2(M)$ and $A'_0 \in H_2(M')$ which evaluate to 1 on the symplectic forms and set

$$\Delta = c_1(A_0), \quad \Delta' = c_1(A'_0).$$

We will use these bases to define the algebras $\mathfrak{A}(X, M)$ and $\mathfrak{A}(X, M')$. Suppose that $d(K) = d(K')$. If $d = 0$, then

$$\begin{aligned} \Delta &= \gcd(\Delta, 0) = d(K) \\ \Delta' &= \gcd(\Delta', 0) = d(K'). \end{aligned}$$

This implies $\deg(q_{k,i}) = \deg(q'_{k,i})$ for all $k \in \mathbb{N}, 0 \leq i \leq a$. Hence the map

$$q_{k,i} \mapsto q'_{k,i}, \quad k \in \mathbb{N}, 0 \leq i \leq a,$$

induces a degree preserving isomorphism $\phi: \mathfrak{A}(X, M) \rightarrow \mathfrak{A}(X, M')$.

Suppose $d \geq 1$. Under our assumptions, the sets Q_b and Q'_b have the same cardinality for each $0 \leq b < d$, cf. Lemma 40 and Example 39. Hence there exists a bijection

$$\psi: \mathbb{N} \times \{0, \dots, a\} \rightarrow \mathbb{N} \times \{0, \dots, a\}, (k, i) \mapsto \psi(k, i),$$

such that

$$\deg(q_{k,i}) \equiv \deg(q'_{\psi(k,i)}) \pmod{2d}.$$

Since z'_1 has degree $-2d$, there exists for each $(k, i) \in \mathbb{N} \times \{0, \dots, a\}$ an integer $\alpha(k, i) \in \mathbb{Z}$, such that

$$\deg(q_{k,i}) = \deg(z_1'^{\alpha(k,i)} q'_{\psi(k,i)}).$$

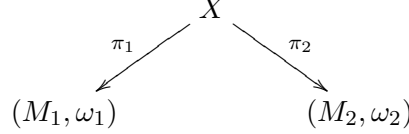
The map

$$q_{k,i} \mapsto z_1'^{\alpha(k,i)} q'_{\psi(k,i)}, \quad k \in \mathbb{N}, 0 \leq i \leq a,$$

therefore induces a well-defined, degree preserving isomorphism $\phi: \mathfrak{A}(X, M) \rightarrow \mathfrak{A}(X, M')$ over $\mathfrak{L}(X)$. \square

Using Theorem 35 and Proposition 28 we get the following corollary. The part concerning equivalent contact structures follows because equivalent contact structures have isomorphic contact homologies.

Corollary 43. *Let X be a closed, simply-connected 5-manifold which can be realized in two different ways as a Boothby-Wang fibration over closed, simply-connected symplectic 4-manifolds (M_1, ω_1) and (M_2, ω_2) , whose symplectic forms represent integral and indivisible classes:*



Denote the associated Boothby-Wang contact structures on X by ξ_1 and ξ_2 and the canonical classes of the symplectic structures by K_1 and K_2 . Let $d(\xi_i)$ denote the divisibility of the first Chern class of ξ_i and $d(K_i)$ the divisibility of K_i . Then:

- *The almost contact structures underlying ξ_1 and ξ_2 are equivalent if and only if $d(\xi_1) = d(\xi_2)$.*

Suppose that ξ_1 and ξ_2 are equivalent as contact structures.

- *If $d(\xi_1) = d(\xi_2) = 0$, then $d(K_1) = d(K_2)$.*
- *If $d(\xi_1) = d(\xi_2) \neq 0$, then either both $d(K_1), d(K_2) \leq 3$ or $d(K_1) = d(K_2) \geq 4$.*

8. APPLICATIONS

In order to apply Corollary 43, it is useful to have as many contact structures on different levels of X as possible. By Lemma 30, the level is always a multiple of the divisibility of the canonical class. We first want to show that one can perturb a single symplectic form ω on a given simply-connected 4-manifold M without changing the canonical class K , so that the induced Boothby-Wang contact structures realize all levels which are non-zero multiples of the divisibility $d(K)$.

Lemma 44. *Let (M, ω) be a minimal closed symplectic 4-manifold with $b_2^+(M) > 1$ and canonical class K . Then every class in $H^2(M; \mathbb{R})$ of the form $[\omega] + tK$ for a real number $t \geq 0$ can be represented by a symplectic form.*

Proof. Note that the canonical class K is a Seiberg-Witten basic class. Since M is assumed minimal, Proposition 3.3 and the argument in Corollary 3.4 in [11] show that K is represented by a disjoint collection of embedded symplectic surfaces in M all of which have non-negative self-intersection. The inflation procedure [15], which can be done on each of the surfaces separately and with the same parameter $t \geq 0$, shows that $[\omega] + tK$ is represented by a symplectic form on M . \square

We can now prove:

Theorem 45. *Let M be a closed, minimal simply-connected 4-manifold such that $b_2^+(M) > 1$ and ω a symplectic form on M . Denote the canonical class of ω by K and let $m \geq 1$ be an arbitrary integer. Then there exists a symplectic form ω' on M , deformation equivalent to ω and representing an integral and indivisible class, such that the first Chern class of the associated Boothby-Wang contact structure ξ' has divisibility $d(\xi') = md(K)$.*

Proof. Let $k = d(K)$. We can assume that ω is integral and choose a basis for $H^2(M; \mathbb{Z})$ such that

$$\begin{aligned} K &= k(1, 0, \dots, 0) \\ \omega &= (\omega_1, \omega_2, 0, \dots, 0). \end{aligned}$$

By a deformation we can assume that ω is not parallel to K , hence $\omega_2 \neq 0$. We can also assume that ω_1 is negative while ω_2 is positive: Consider the change of basis vectors

$$\begin{aligned} (1, 0, 0, \dots, 0) &\mapsto (1, 0, 0, \dots, 0) \\ (0, 1, 0, \dots, 0) &\mapsto (q, \pm 1, 0, \dots, 0), \end{aligned}$$

where q is some integer. Then the expression of ω in the new basis is

$$(\omega_1 \mp q\omega_2, \pm\omega_2, 0, \dots, 0).$$

Hence if q is large enough, has the correct sign and the \pm sign is chosen correctly, the claim follows.

Suppose that $\sigma \in H^2(M; \mathbb{Z})$ is an indivisible class of the form

$$\sigma = (\sigma_1, \sigma_2, 0, \dots, 0)$$

which can be represented by a symplectic form, also denoted by σ . Let ζ denote the contact structure induced on the Boothby-Wang total space by σ . We claim that the divisibility $d(\zeta)$ is given by

$$d(\zeta) = k|\sigma_2|.$$

To prove this we write $K = -c_1(M) = rR + \gamma\sigma$, where $R = (R_1, R_2, 0, \dots, 0)$. Then $k - \gamma\sigma_1$ and $\gamma\sigma_2$ are divisible by r . This implies that r divides $k\sigma_2$. Conversely note that by assumption σ_1, σ_2 are coprime. Let R_1, R_2 be integers with

$$1 = \sigma_2 R_1 - \sigma_1 R_2$$

and define

$$\gamma = -kR_2.$$

Then we can write

$$K = k\sigma_2 R - kR_2 \sigma.$$

This proves the claim about $d(\zeta)$.

Suppose that $m \geq 1$. By multiplying the expression for ω with the positive number $\frac{m}{\omega_2}$ we see that the (rational) class

$$(\alpha, m, 0, \dots, 0), \quad \alpha = \omega_1 \frac{m}{\omega_2},$$

is represented by a symplectic form. Note that $\alpha < 0$. By the inflation trick in Lemma 44 with parameter $t = \frac{1}{k}(1 - \alpha)$ it follows that

$$\begin{aligned} \omega' &= (\alpha, m, 0, \dots, 0) + (1 - \alpha, 0, \dots, 0) \\ &= (1, m, 0, \dots, 0) \end{aligned}$$

is represented by a symplectic form ω' . The class ω' is indivisible. Let ξ' denote the induced Boothby-Wang contact structure. By our calculation above, $d(\xi') = mk$. \square

Definition 46. For integers $d \geq 4$ and $r \geq 2$ denote by $Q(r, d)$ the number of integers in the following set:

$$\left\{ k \in \mathbb{N} \left| \begin{array}{l} k \geq 4, k \text{ divides } d \text{ and there exists a simply-connected} \\ \text{symplectic 4-manifold } (M, \omega) \text{ with } b_2(M) = r \text{ and } b_2^+(M) > 1 \\ \text{whose canonical class } K \text{ has divisibility } d(K) = k. \end{array} \right. \right\}$$

The numbers $Q(r, d)$ are connected to the geography of simply-connected symplectic 4-manifolds with divisible canonical class. The following lemma relates knowledge about the numbers $Q(r, d)$ to the existence of inequivalent contact structures on simply-connected 5-manifolds. Here we make essential use of Corollary 43 and Theorem 45.

Lemma 47. *Let $d \geq 4$ and $r \geq 2$ be integers. Suppose that either*

- *d is odd and X the simply-connected 5-manifold $(r-2)S^2 \times S^3 \# S^2 \tilde{\times} S^3$,*
- or*
- *d is even and X the simply-connected 5-manifold $(r-1)S^2 \times S^3$.*

In both cases, there exist at least $Q(r, d)$ many inequivalent contact structures on the level d on X .

Proof. Recall that a spin (non-spin) simply-connected 5-manifold has only even (odd) levels. Suppose that $d \geq 4$ is an integer and (M, ω) a simply-connected symplectic 4-manifold with $b_2(M) = r$ and $b_2^+(M) \geq 2$ whose canonical class has divisibility $k = d(K) \geq 4$ dividing d . We can write $d = mk$. Since the divisibility of K is greater than 1, the manifold M is minimal, because non-minimal symplectic 4-manifolds contain a symplectically embedded 2-sphere S with intersection number $K \cdot S = -1$. By Theorem 45 there exists a symplectic structure ω' on M that induces on the Boothby-Wang total space X with $b_2(X) = r-1$ a contact structure with $d(\xi) = d$. Since the symplectic form ω' is deformation equivalent to ω the canonical class K remains unchanged. By Corollary 43 the contact structures on the same non-zero level d on X coming from symplectic 4-manifolds with different divisibilities $k \geq 4$ of their canonical classes are pairwise inequivalent. \square

We define the following purely number theoretic numbers.

Definition 48. For an integer $d \geq 4$ let $N(d)$ denote the number of positive integers ≥ 4 dividing d . If d is even, let $N'(d)$ denote the number of odd divisors ≥ 4 of d .

The following lemma gives a bound on the maximal number of inequivalent contact structures that can be distinguished with our method.

Lemma 49. *Let $d \geq 4$ and $r \geq 2$ be integers. Then there are the following upper bounds for $Q(r, d)$.*

- (a) *For any r we have $Q(r, d) \leq N(d)$.*
- (b) *If d is even and r is not congruent to 2 mod 4, then $Q(r, d) \leq N(d')$.*

Proof. The first statement is clear by the definitions. For the second statement, suppose that M is a simply-connected symplectic spin 4-manifold. Then the intersection form Q_M is even and $b_2^+(M)$ odd. Note that $b_2^- = b_2^+ - \sigma$, hence

$b_2(M) = 2b_2^+(M) - \sigma(M)$. Since Q_M is even, the signature $\sigma(M)$ is divisible by 8. This implies that $b_2(M)$ is congruent to 2 mod 4 because $b_2^+(M)$ is odd for a simply-connected symplectic 4-manifold. Hence if r is *not* congruent to 2 mod 4 then there does not exist a simply-connected symplectic spin 4-manifold M with second Betti number r . Hence all elements of $Q(r, d)$ are in this case odd. \square

To calculate some of the numbers $Q(r, d)$ we can use the geography work in [10]. For example, recall that a homotopy elliptic surface M is a closed, simply-connected 4-manifold that is homeomorphic to a surface of the form $E(m)_{p,q}$ with p, q coprime. By definition, homotopy elliptic surfaces have topological invariants

$$c_1^2(M) = 0, \chi_h(M) = m, b_2(M) = 12m - 2, b_2^+(M) = 2m - 1.$$

In [10, Theorem 15] we proved the following:

Theorem 50. *Let m and k be positive integers. If m is odd, assume that k is odd also. Then there exists a symplectic homotopy elliptic surface M with $\chi_h(M) = m$ whose canonical class K has divisibility k .*

This implies the following proposition about some of the numbers $Q(r, d)$:

Proposition 51. *Let $n \geq 1$ and $d \geq 4$ be arbitrary integers.*

- (a) *If d is odd, then $Q(12n - 2, d) = N(d)$.*
- (b) *If d is even, then $Q(24n - 2, d) = N(d)$ and $Q(24n - 14, d) \geq N'(d)$.*

Proof. For part (a), let $r = 12n - 2$ and suppose that $d \geq 4$ is odd. To prove the claim, we have to find for every divisor $k \geq 4$ of d a simply-connected 4-manifold M with $b_2 = r$ and $b_2^+ > 1$ whose canonical class has divisibility k . Since d is odd, the integer k is odd also. If $n > 1$, there exists by Theorem 50 a symplectic homotopy elliptic surface M with $b_2(M) = r$, $b_2^+(M) \geq 3$ and $d(K) = k$. If $n = 1$ we choose as M a Dolgachev surface with $b_2(M) = 10$, $b_2^+(M) = 1$ and $d(K) = k$. Since the canonical class of a Dolgachev surface is represented by two disjoint tori of self-intersection zero, given by the multiple fibres, the proofs of Lemma 44 and Theorem 45 also work in this case even though $b_2^+ = 1$.

To prove part (b), suppose that $d \geq 4$ is even and let $r = 24n - 2$. Then for every divisor $k \geq 4$ of d there exists by Theorem 50 a symplectic homotopy elliptic surface M with $b_2(M) = r$ and $d(K) = k$. This proves the first claim. Let $r = 24n - 14 = 12(2n - 1) - 2$. Then for every odd divisor $k \geq 4$ of d there exists by Theorem 50 a symplectic homotopy elliptic surface M with $b_2(M) = r$ and $d(K) = k$. This proves the second claim. \square

As a corollary we get with Lemma 47 the following result about the existence of inequivalent contact structures in the same equivalence class of almost contact structures:

Corollary 52. *Let $n \geq 1$ be an arbitrary integer.*

- (a) *On every odd level $d \geq 5$ the 5-manifold $(12n - 4)S^2 \times S^3 \# S^2 \tilde{\times} S^3$ admits at least $N(d)$ inequivalent contact structures.*
- (b) *On every even level $d \geq 4$ the 5-manifold $(24n - 3)S^2 \times S^3$ admits at least $N(d)$ inequivalent contact structures.*

- (c) *On every even level $d \geq 4$ the 5-manifold $(24n - 15)S^2 \times S^3$ admits at least $N'(d)$ inequivalent contact structures.*

In a similar way we can use other geography results from [10] to find more inequivalent contact structures on the same level on simply-connected 5-manifolds X of the form $rS^2 \times S^3$ and $rS^2 \times S^3 \# S^2 \tilde{\times} S^3$.

REFERENCES

1. D. Barden, *Simply connected five-manifolds*, Ann. of Math. **82**, 365–385 (1965).
2. W. M. Boothby, H. C. Wang, *On contact manifolds*, Ann. of Math. **68**, 721–734 (1958).
3. F. Bourgeois, *A Morse-Bott approach to contact homology*, Ph. D. Thesis, Stanford 2002.
4. H. Duan, C. Liang, *Circle bundles over 4-manifolds*, Arch. Math. (Basel) **85**, 278–282 (2005).
5. Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal., Special Volume, Part II, 560–673 (2000).
6. H. Geiges, *Contact structures on 1-connected 5-manifolds*, Mathematika **38**, 303–311 (1991).
7. R. E. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142**, 527–595 (1995).
8. J. W. Gray, *Some global properties of contact structures*, Ann. of Math. **69**, 421–450 (1959).
9. M. J. D. Hamilton, *On symplectic 4-manifolds and contact 5-manifolds*, Ph. D. thesis, Munich 2008, <http://edoc.ub.uni-muenchen.de/8779/>
10. M. J. D. Hamilton, *The geography of symplectic 4-manifolds with divisible canonical class*, Pacific Journal of Mathematics **244**, 37–84 (2010).
11. M. J. D. Hamilton, D. Kotschick, *Minimality and irreducibility of symplectic four-manifolds*, Intern. Math. Res. Notices 2006, Article ID 35032, 1–13.
12. S. Kobayashi, *Principal fibre bundles with the 1-dimensional toroidal group*, Tohoku Math. J. **8**, 29–45 (1956).
13. O. van Koert, *Open books for contact five-manifolds and applications of contact homology*, Ph. D. Thesis, Köln 2005.
14. O. van Koert, *Open books on contact five-manifolds*, Annales de l’institut Fourier **58**, 139–157 (2008).
15. F. Lalonde, D. McDuff, *The classification of ruled symplectic 4-manifolds*, Math. Res. Lett. **3**, 769–778 (1996).
16. D. McDuff, D. Salamon, *Introduction to symplectic topology*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
17. J. Milnor, J. Stasheff, *Characteristic classes*, Princeton University Press 1974.
18. J. Pati, *Contact homology of S^1 -bundles over some symplectically reduced orbifolds*, preprint arXiv:0910.5934
19. I. Ustilovsky, *Infinitely many contact structures on S^{4m+1}* , Internat. Math. Res. Notices **14**, 781–791 (1999).

E-mail address: mark.hamilton@math.lmu.de